Exam Waves and Optics – 30 January 2014

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Questions and answers

A few **preliminary remarks**:

- Answers may be given in Dutch.
- Use a new sheet of paper for each question.
- Put your name and student number at the top of all sheets.
- Put your student card at the edge of the desk for checking by the assistants and show it when handing in your exam.

Question 1 (10 points). Energy transport by an electromagnetic wave

The energy density (unit: J/m^3) of an electric field of magnitude E, in vacuum, is given by:

$$u_E = \frac{\varepsilon_0}{2} E^2$$

The energy density (unit: J/m^3) of a magnetic field of magnitude B, in vacuum, is given by:

$$u_B = \frac{1}{2\mu_0} B^2$$

(ε_0: electric permittivity of the vacuum; μ_0: magnetic permeability of the vacuum)

Questions:

a. Show, based on the properties of an electromagnetic wave, that the energy densities of the electric and magnetic field associated with an electromagnetic wave are equal: $u_E = u_B$.

b. Derive an expression for the so-called Poynting-vector which describes the direction and magnitude of the (instantaneous) transport of energy per unit time and unit surface area, expressed in J/(s m^2) or W/m^2, of an electromagnetic wave.
Hint: Consider the energy flowing during a time interval Δt through a surface area A.

Answer

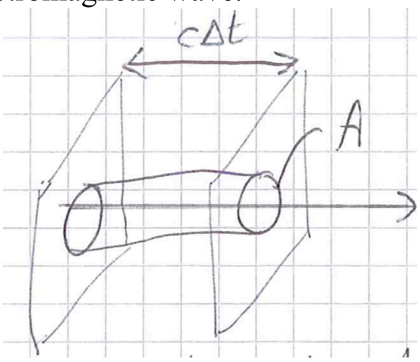
a. using the following properties of an electromagnetic wave:

$B = \frac{E}{c}$ and $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$ or $\frac{1}{\mu_0} = \varepsilon_0 c^2$, one gets:

$$u_B = \frac{1}{2}\varepsilon_0 c^2 \frac{1}{c^2} E^2 = \frac{1}{2}\varepsilon_0 E^2 = u_E$$

(c is the speed of light in vacuum)

b. The sketch below is used. The arrow indicates the direction of propagation of an electromagnetic wave.



The energy transported through the surface area A in the time Δt is equal to the energy density integrated over the cylindrical volume "$A \times c\Delta t$".
Considering the magnitude of the (instantaneous) transport of energy per unit time and unit surface area (S), the time interval Δt can be considered much smaller than the period of the electromagnetic wave, such that E and B can be considered constant during Δt. We then have:

$$S = \frac{u\ A\ c\ \Delta t}{A\ \Delta t} = u\ c$$

Using the various expressions for u, this leads e.g. to:

$$S = \frac{1}{\mu_0} B^2 = \frac{1}{\mu_0} E\ B = c^2 \varepsilon_0 E\ B$$

Considering the direction of energy transport, it is reasonable to assume that energy is transported in the direction of propagation of the wave. As the direction of propagation of an electromagnetic wave is given by the direction of $\overline{E} \times \overline{B}$, the vector

$$\overline{S} = \frac{1}{\mu_0} \overline{E} \times \overline{B} = c^2 \varepsilon_0 \overline{E} \times \overline{B}$$

(the so-called Poynting vector) has the direction and magnitude of the (instantaneous) transport of energy per unit time and unit surface area of an electromagnetic wave.

Question 2 (6 points): Wave functions

Which of the following expressions represents a wave function ? Explain how you reach your conclusion.

a) $\psi(x,t) = (2x - 3t)^2$

b) $\psi(x,t) = (6x + 2t + 5)^2$

c) $\psi(x,t) = 1/(x^2 + 3t)$

x represents the spatial coordinate (unit: *m*)
t represents time (unit: *s*)

For those representing a wave function, what is the size and direction of the velocity of the wave ?

Answer:

a) and b) are wave functions as they can be written in the form $f(x \pm vt)$, with v the velocity of the wave; c) can not be written in this form and is thus not a wave function. Alternatively, one can show that a) and b) fulfil the one-dimensional differential wave equation, whereas c) does not.

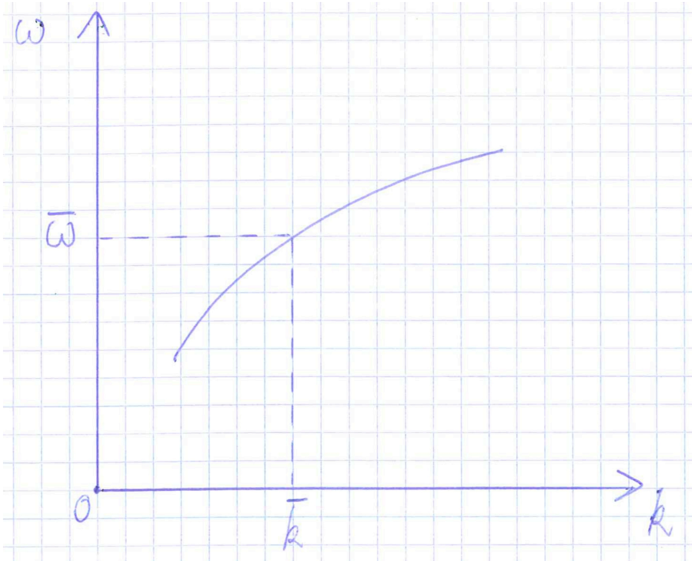
The velocity is obtained after rewriting the wave function:

a) $\psi(x,t) = (2(x - 3/2t))^2$ and thus $v = 3/2 = 1.5$ *m/s*
moving in the positive x-direction

b) $\psi(x,t) = (6(x + 2/6t) + 5)^2$ and thus $v = 2/6 = 0.33$ *m/s*
moving in the negative x-direction

Question 3 (4 points): The dispersion relation

In the description of the superposition of harmonic waves with different frequencies, the concepts of phase velocity and group velocity are used. The dispersion relation relates the angular frequency ω of a wave to its propagation number k: $\omega(k)$. Making use of the dispersion relation, a graphical representation of the phase and group velocities was discussed during the lectures. Below an example of a dispersion relation.



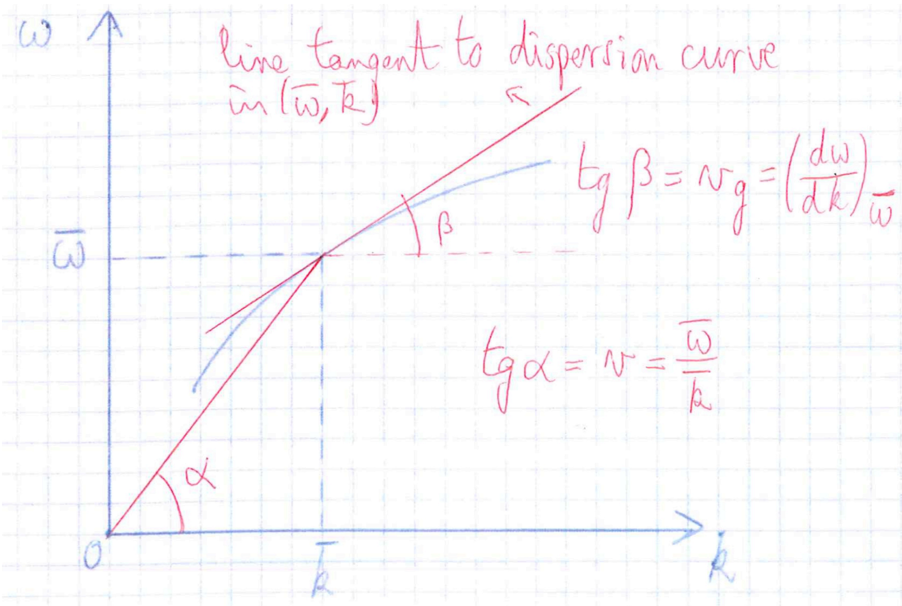
Question:

Indicate in this figure (make a copy of it in your answer) the graphical representation of the phase velocity and the group velocity of a superposition of harmonic waves with average angular frequency $\bar{\omega}$ and average propagation number $\bar{k}$.

Answer:

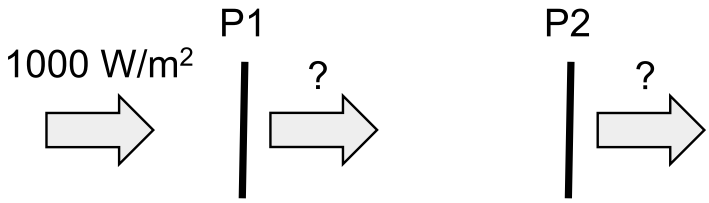
phase velocity: $v = \frac{\varpi}{\bar{k}}$ group velocity: $v_g = \left(\frac{d\omega}{dk}\right)_{\varpi}$

Graphically, v is the tangent of the slope of the line through $(\varpi, \bar{k})$ and the origin and v_g is the tangent of the slope of the line that is tangent to the dispersion curve in the point $(\varpi, \bar{k})$. Added to the figure:



Question 4 (6 points): Linear polarizers

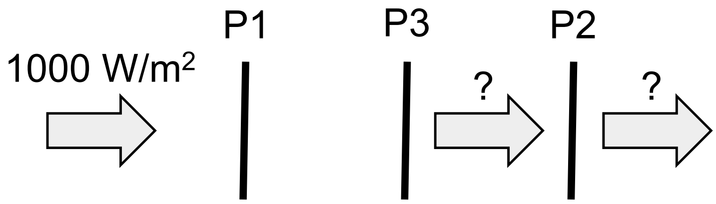
Natural light with an irradiance of 1000 W/m^2 impinges on two consecutive linear polarizers P1 and P2 (see figure below).



Polarizer P1 has a direction of polarization of +20 degrees relative to the vertical, polarizer P2 has a direction of polarization of +50 degrees relative to the vertical.

Question a:
What is the irradiance in W/m^2 of the light after P1 and after P2 ?

Now consider that a third linear polarizer P3, with a direction of polarization of –25 degrees relative to the vertical, is placed between P1 and P2 (see figure below).



Question b:
What is the irradiance in W/m^2 of the light after P3 and after P2 ?

Note: The polarization direction relative to the vertical is expressed by a positive or negative number. The '+' and '–' sign indicate opposite directions relative to the vertical: clockwise and counterclockwise.

Answer

Question a: A linear polarizer transmits 50% of the irradiance of natural light. So after P1, 500 W/m^2 remains. For the transmission of the light (linearly polarized after P1) through P2, we use Malus's Law. The angle between the polarization directions of P1 and P2 is 50 – 20 = 30 degrees. So the irradiance after P2 is:

$$I_{P2} = 500 \times \cos^2 30 = 500 \times 0.75 = 375 \text{ W/m}^2$$

Question b: The angle between the polarization directions of P3 and P1 is -25 – 20 = -45 degrees. So the irradiance after P3 is:

$$I_{P3} = 500 \times \cos^2(-45) = 500 \times 0.5 = 250 \text{ W/m}^2$$

The angle between the polarization directions of P2 and P3 is 50 – (-25) = 75 degrees. So the irradiance after P2 is:

$$I_{P2} = 250 \times \cos^2 75 = 500 \times 0.067 = 16.7 \text{ W/m}^2$$

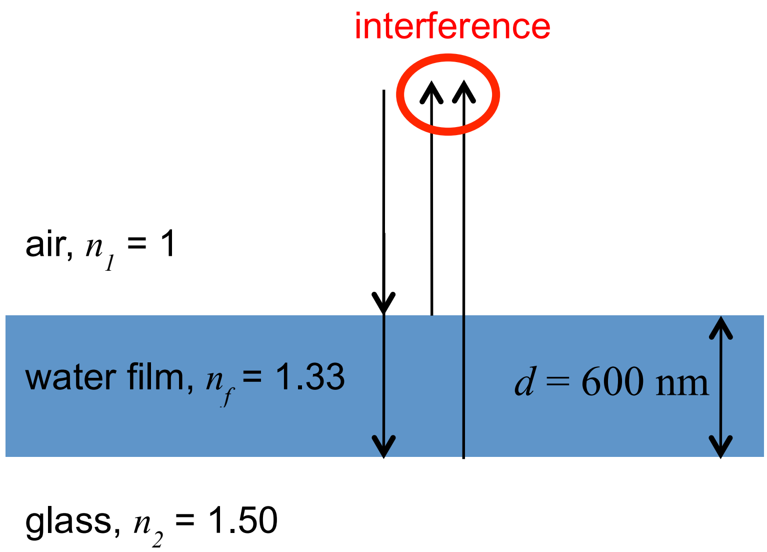
Question 5 (8 points): Reflection from a thin film

Consider a layer of water (index of refraction 1.33) with a thickness of 600 nm deposited on a glass plate (index of refraction 1.50). Above the water is air. The film is illuminated from above with sunlight.

Question:
Which vacuum wavelengths within the visible spectrum (300 to 800 nm) are suppressed due to interference in the reflected light ?

Answer

The situation is sketched below.



External reflection happens at both the air-water and water-glass interface. The two interfering beams thus do not show a relative phase shift due to the reflections because the reflections are of the same type.
A phase shift is introduced due to the difference in optical path length. If this phase shift is $(1+2m)\pi$ radians (m an integer number), the interfering beams are 180 degrees out of phase and will suppress each other maximally.
The phase shift due to the difference in optical path length (ΔOPL) is:

$\Lambda = k_0\ \Delta\text{OPL}$, with k_0 the propagation number of the light in vacuum

$\Delta\text{OPL} = 2\ n_f\, d$ (factor 2 because the film is traversed twice)

$k_0 = 2\pi/\lambda_0$ (λ_0 is the vacuum wavelength)

putting all of this together:

$$(1+2m)\pi = \frac{2\pi}{\lambda_0} 2n_f d \quad \text{or} \quad \lambda_0 = \frac{4n_f d}{1+2m} = \frac{4\times1.33\times600}{1+2m} = \frac{3192}{1+2m}\ \text{nm}$$

λ_0 falls within the 300 to 800 nm wavelength range for:

$m = 2$ ($\lambda_0 = 638.4$ nm)

$m = 3$ ($\lambda_0 = 456$ nm)

$m = 4$ ($\lambda_0 = 354.7$ nm)

Question 6 (6 points): The Huygens-Fresnel principle

In the lectures, the Huygens-Fresnel principle was used to deduce/study the propagation of waves and their diffraction at apertures. The Huygens-Fresnel principle states:

> Every unobstructed point of a wavefront, at a given instant, serves as a source of spherical secondary wavelets (with the same frequency as the primary wave). The amplitude of the optical field at any point beyond is the superposition of all these wavelets.

The Huygens-Fresnel principle as stated above has a shortcoming that is solved by the introduction of the inclination (or obliquity) factor.

Questions:

a) Which problem of the Huygens-Fresnel principle does the inclination factor solve ?

b) What is the inclination factor (explain in words and illustrate with a drawing) ?

Even taking the inclination factor into account, a remaining shortcoming of the Huygens-Fresnel principle was pointed out during the lectures.

Question:

c) Which shortcoming of the Huygens-Fresnel principle remains after taking the inclination factor into account ?

Answer

a) Without the inclination factor, the superposition of the secondary waves in the backward direction leads to a wave propagating in the backwards direction. Such a wave is not observed in nature. The inclination factor ensures that only a forward propagating wave follows from the Huygens-Fresnel principle.

b) The inclination (or obliquity) factor is a modification to the amplitude of the secondary waves. The amplitude is maximal (inclination factor 1) in the same propagation direction as the primary wave, decreases with increasing angle with respect to the propagation direction of the primary wave and becomes zero for the secondary wave propagating in the opposite direction of the primary wave. See the drawing below.

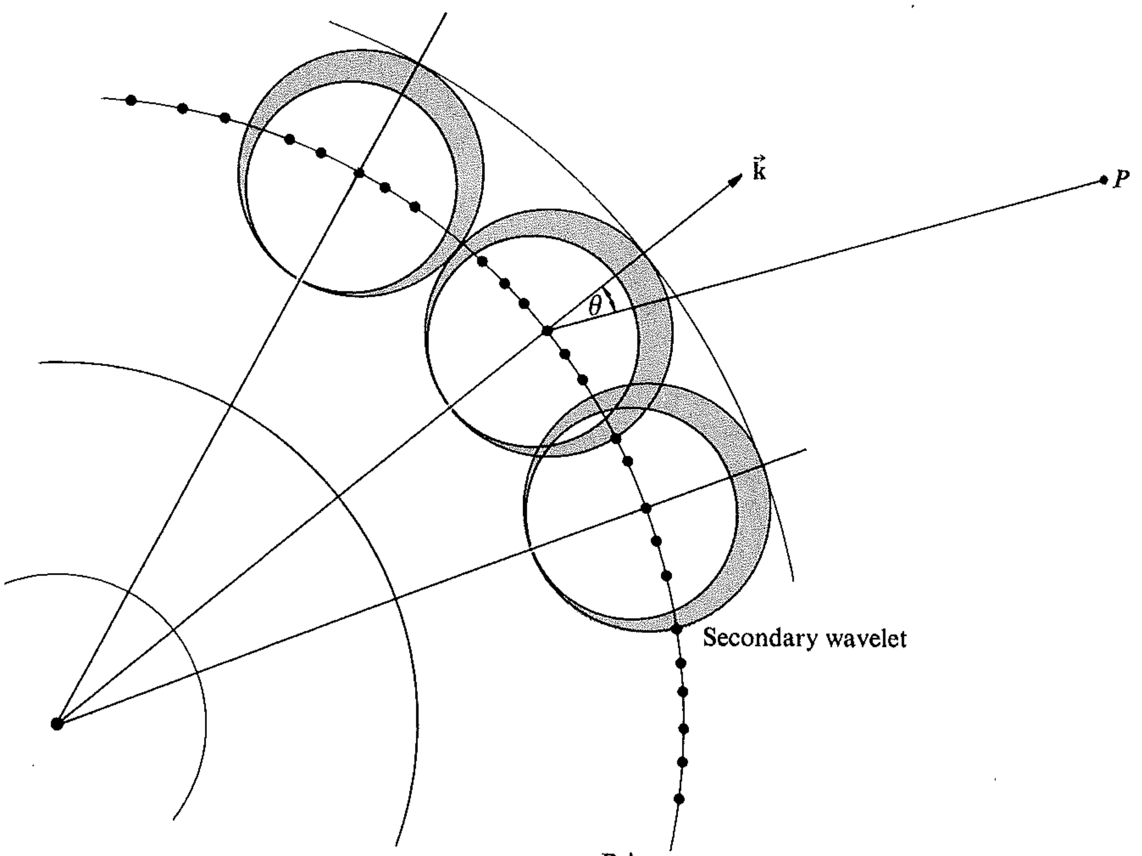


Figure 10.36 Secondary wavelets.

c) Comparing the direct propagation of a wave with the propagation according to the Huygens-Fresnel principle shows a π/2 phase shift. The shortcoming of the Huygens-Fresnel principle is thus the lack of a π/2 phase shift between the primary and secondary waves.
